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Relaxation of superflow in a network: Application to the dislocation model of supersolidity of helium crystals

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We have considered the dislocation network model for the supersolid state in ^{4}He crystals. In difference with uniform two-dimensional and three-dimensional systems, the temperature of superfluid transition T_c in the network is much smaller than the degeneracy temperature T_d. It is shown that a crossover into a quasisuperfluid state occurs in the temperature interval between T_c and T_d. Below the crossover temperature the time of decay of the flow increases exponentially under decrease in the temperature. The crossover has a continuous character and the crossover temperature does not depend on the density of dislocations.

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Forty years ago Andreev and Lifshitz[1] predicted the possibility of realization of a ground state of quantum crystals with the number of sites larger than the number of atoms. Empty sites in such a state are called zero-point vacancies. In crystals consisting of Bose atoms zero-point vacancies behave as Bose quasiparticles, and at low temperatures they become superfluid ones. Therefore the phenomenon predicted by Andreev and Lifshitz is called the supersolids. In 1970s the problem of supersolids addressed a number of theoretical papers,[2–6] but numerous experimental efforts to discover this phenomenon failed (see Ref. 7). The situation has changed completely after the experiment of Kim and Chan[8] in which a nonclassical rotational inertia (NCRI) of ^{4}He crystal was observed below certain critical temperature. NCRI means that the ^{4}He crystal does not rotate as a rigid body, but certain (superfluid) fraction of atoms is decoupled from the rotation. Further experiments[9–13] confirmed the NCRI effect. It was established[9–14] that the NCRI is not an intrinsic property of ^{4}He crystals. The amount of fraction decoupled from the oscillations (rotation) depends considerably on the degree of disorder in the crystal lattice; in particular, it becomes much smaller after annealing. A correlation between supersolid features and disorder was also observed in the direct flow experiment.[15] A common explanation of the correlation between disorder and supersolid behavior is that zero-point vacancies may emerge only at extended crystal lattice defects, presumable, in dislocation cores. The idea on superfluid behavior of dislocations was put forward in Ref. 16 (originally, in a context of description of anomalous plastic properties of parahydrogen[17]). In Ref. 18 this idea was applied for the explanation of NCRI. In Ref. 18 and in the following[19] papers superfluid properties of dislocations were demonstrated by the first-principles Monte Carlo calculations. A combined (dislocation network plus bulk) mechanism of supersolidity was considered in Ref. 20. The role of dislocations in this mechanism is similar to one for the enhancement of superconductivity by dislocations.[21]

In this Rapid Communication we study the specifics of the superfluid transition in the network of one-dimensional (1D) wires and arrive at the following conclusion. The temperature of the superfluid transition in the network is quite small and it depends on the length of segments of the network. At the same time, a crossover into a quasisuperfluid state (a state with exponentially large time of relaxation of a flow) takes place at much larger temperatures, and the crossover temperature does not depend on the length of the segments. It is just the behavior observed in torsion experiments where the transition is continuous, and it occurs in the temperature interval independent of the NCRI fraction.

The problem of the superfluid transition in a dislocation network was studied in Ref. 22. It was shown that in the two-dimensional (2D) network the critical temperature is proportional to the inverse length of the segment. Let us derive the result of Ref. 22 using the following simple arguments. For a 2D uniform medium the Berezinskii-Kosterlitz-Thouless (BKT) critical temperature is determined by the equation $T_c = \pi\hbar^2 n_{s2}(T_c)/2m$, where n_{s2} is the two-dimensional superfluid density. The BKT transition is connected with the energy of a vortex pair diverges logarithmically at large distances between the vortices. At such distances the difference between a uniform medium and a network is not important. Therefore, the critical temperature for the network is given by the BKT equation in which the uniform density n_{s2} is replaced with the average 2D superfluid density $\bar{n}_2$. The latter quantity can be expressed through the 1D density of zero-point vacancies n and the length of the segment of the network l. For the quadratic network $\bar{n}_2 = 2n/l$, and the critical temperature is $T_c = \pi\hbar^2 n/ml$ (m is the effective mass of zero-point vacancies).

In Ref. 23 a microscopic model of Bose-Einstein condensation (BEC) of noninteracting zero-point vacancies in a three-dimensional (3D) network was considered. It was shown that the BEC temperature depends significantly on the transparency of vertices (intersections of dislocations), and the highest value, reached at high transparency, is $T_{\mathrm{BEC}} \approx 3\hbar^2 n/2ml$ so the superfluid critical temperatures in 3D and 2D networks are of the same order.

The critical temperature T_c in a network is much smaller than the degeneracy temperature for the 1D Bose gas of vacancies in a segment $T_d = \hbar^2 n^2/m$ [the small parameter is $(nl)^{-1} \ll 1$]. In this respect the network differs from uniform 2D and 3D systems, where $T_c \approx T_d$. One can expect that the

network may demonstrate quasisuperfluid behavior in a wide range of temperatures between T_c and T_d.

To be more specific we consider a 2D network. In a uniform 2D system the vortex-antivortex pairs unbind above the BKT transition and vortices of opposite vorticities can move independently from each other. If a given vortex crosses the system in a direction perpendicular the flow, the superfluid phase difference along the flow changes on 2π. In a uniform system a motion of a vortex across the flow is caused by the Magnus force and the viscous friction between the vortex and the normal component. The network is a multiple connected system, the vortices correspond to circular currents, and they are pinned to given plaquettes. The vortex centers cannot move freely, but they can jump from one plaquette to another. This process becomes possible due to an emergence of phase slip (PS) centers at the segments.

The theory of dissipation of supercurrent in 1D channels based on the idea of emergence of PS centers was put forward by Langer and Ambegaokar.[24] The main shortage of the theory[24] is that it does not yield the pre-exponential factor in the expression for the relaxation time. To obtain this factor one should consider the dynamics of transition of the system over the potential barrier under appearance of a PS center. This problem was solved in Ref. 25 on the base of the *diffusive* (time-dependent Ginzburg-Landau) equation. We solve the problem with the use of the *wave* (Gross-Pitaevskii) equation that describes an essentially different physical mechanism of relaxation of the supercurrent.

If the vortex is already present in a given plaquette the PS of the proper sign provides annihilation of that vortex and creation of a vortex of the same vorticity in the neighbor plaquette. A vortex may jump to any neighbor plaquette, but there is a preferable direction of such jumps in a system with a flow. Let us, for simplicity, consider a regular quadratic network with a flow directed parallel to the segments (say x direction). If the vortex is centered in a given plaquette the superfluid velocities in the segments that form this plaquette read as $v_A=v_v+v_s$, $v_C=v_v-v_s$, $v_B=v_D=v_v$, where A and C stand for the segments oriented along the x axis, while B and D—for the segments oriented along the y axis (perpendicular to the flow). Here v_s is the flow velocity, and $v_v=\pi\hbar/2ml$ is the contribution caused by the vortex. The frequency of PS in a segment is proportional to its length, and it is a function of the superfluid velocity $\nu_i=lf(v_i)$. One can see that $\nu_B=\nu_D$ and in average the vortices do not move in the x direction (the direction of the flow). The PS frequencies for two other segments differ from each other and one extra jump in the perpendicular to the flow direction gains with the frequency

$$\Delta\nu=\nu_A-\nu_C=\alpha l v_s, \tag{1}$$

where $\alpha=2f'(v)|_{v=0}$ (here we imply the limit of small superfluid velocities). In the network of a rectangle shape of area $S=L_x\times L_y$ a vortex crosses the system with the frequency $\nu_{cross}=N_v\Delta\nu l/L_y$, where $N_v=n_vS$ is the total number of unbound vortices (n_v is the vortex density). Each cross lowers the phase gradient on $2\pi/L_x$ that changes the flow velocity on $\Delta v_s=-2\pi\hbar/mL_x$. The equation for v_s, written in the differential form, reads as

$$\frac{dv_s}{dt}=\nu_{cross}\Delta v_s=-\frac{2\pi\hbar\alpha n_v l^2}{m}v_s. \tag{2}$$

The solution of Eq. (2) is $v_s=v_{s0}e^{-t/\tau}$, where $\tau=m/2\pi\hbar\alpha n_v l^2$ is the decay time. To compute α one should specify the mechanism of PS. We will describe the gas of zero-point vacancies in the dislocation core as a weakly nonideal 1D Bose gas with a complex condensate wave function (order parameter) $\Psi(x,t)$ that satisfies the Gross-Pitaevskii (GP) equation. The 1D GP equation with a repulsive point interaction has an exact solution that corresponds to a dark soliton. The dark soliton is a rarefaction that moves with a constant velocity u.

The dark soliton is described by the function[26]

$$\Psi(x,t)=\sqrt{\tilde{n}}\left[\sqrt{1-\frac{u^2}{c^2}}\tanh\left(\sqrt{1-\frac{u^2}{c^2}}\frac{x-ut}{\xi}\right)+i\frac{u}{c}\right]. \tag{3}$$

Here $\xi=\hbar/mc$ is the coherence length, $c=\sqrt{\gamma n/m}$ is the sound velocity, γ is the interaction constant, $\tilde{n}=n[1-2(\xi/l)\sqrt{1-u^2/c^2}]^{-1}$ is the renormalized density (renormalization is the consequence of the conservation of the total number of zero-point vacancies). The energy of the soliton reads as

$$E_0=\int_0^l dx\left[\frac{\hbar^2}{2m}\left(\frac{d\Psi}{dx}\right)^2+\frac{\gamma}{2}(|\Psi|^4-n^2)\right]=\frac{4}{3}\hbar nc\left(1-\frac{u^2}{c^2}\right)^{3/2}. \tag{4}$$

The soliton momentum p can be found by integration of equation $dp=dE_0/u$:

$$p=-2\hbar n\left(\frac{u}{c}\sqrt{1-\frac{u^2}{c^2}}+\arcsin\frac{u}{c}\right)+C, \tag{5}$$

where C is the constant of integration. To determine C one can take into account that the soliton momentum is the difference of the momentum of the system with and without the soliton. The soliton emerges with the velocity $u=+c-0$ or $u=-c+0$, and at such u its momentum should be equal to zero. One can see that two conditions $p_{u=\pm c}=0$ yield two different integration constants $C_\pm=\pm\hbar n\pi$. Therefore, one should consider two species of the solitons (the "+" and "−" ones) with the momenta $p_\pm$ defined by Eq. (5) with $C=C_\pm$.

It is important to emphasize that these two species correspond two physically distinct solitons. According to Eq. (3) the phase at the soliton has the additional shift $\Delta\varphi(u)=-2\cot^{-1}(u/\sqrt{c^2-u^2})$. In any multiple connected system the phase satisfies the Onsager-Feynman quantization condition, and the appearance of a soliton should be accompanied by a change in the net velocity: $v=v_0+\Delta v_\pm(u)$. The function $\Delta\varphi(u)$ is discontinuous at $u=0$ with the jump equal 2π. The functions $\Delta v_\pm(u)$ should be continuous because small variation in u cannot result in a finite change in the net velocity. Since $\Delta v_+(+c)=0$ and $\Delta v_-(-c)=0$, the function $\Delta v_+(u)\neq\Delta v_-(u)$. For instance, for a 1D ring with the perimeter l we find $\Delta v_\pm=\hbar[\pm\pi-2\arcsin(u/c)]/ml$ (we take into account

that $\Delta\varphi + ml\Delta v_{\pm}/\hbar = 0 \bmod 2\pi$). Thus, solitons of distinct species differ from each other by the change in the net velocity they induce.

The soliton may change its velocity due to the interaction with phonons and impurities. Since the soliton changes the net velocity in a continuous way, it is more consistent to consider PS as an entire process of creation of the soliton at $u=+c$ and its annihilation at $u=-c$ or vise versa. The first possibility corresponds to the "+" solitons, and the second one – to the "–" ones.

To obtain the frequency of PS we consider solitons as classical particles whose distribution functions $f_{\pm}(p,t)$ satisfy the Fokker-Planck equation

$$\frac{\partial f_{\pm}}{dt} = -\frac{\partial s_{\pm}}{\partial p}, \tag{6}$$

where $s_{\pm}$ are the soliton fluxes in the momentum space. They read as

$$s_{\pm} = A_{\pm} f_{\pm} - B_{\pm}\frac{\partial}{\partial p} f_{\pm}. \tag{7}$$

The coefficients $A_{\pm}$ and $B_{\pm}$ satisfy the relation $A_{\pm} f_{0,\pm} - B_{\pm}\partial f_{0,\pm}/\partial p = 0$, where $f_{0,\pm} = \exp(-E_{\pm}/T)$ is the equilibrium distribution function, and $E_{\pm} = E_0 + p_{\pm} v$ are the soliton energies at nonzero net velocity. Here we consider the case of slow relaxation. In this case one can neglect the explicit time dependence of $f_{\pm}$ and consider the fluxes $s_{\pm}$ as constant quantities. Using the relation between the coefficient A and B we rewrite Eq. (7) in the form

$$s_{\pm} = -B_{\pm} f_{0,\pm}\frac{\partial}{\partial p}\left(\frac{f_{\pm}}{f_{0,\pm}}\right). \tag{8}$$

The distribution of the solitons with small p is close to equilibrium one. The solitons with $p_{\pm} \to \pm 2\pi\hbar n$ emerge only due to nonzero $s_{\pm}$. To attain such a momentum the soliton should overcome the energy barrier $\Delta E = 4\hbar nc/3$. Therefore the fluxes are small and the distribution functions $f_{\pm}$ at $p = \pm 2\pi\hbar n$ are much less than the equilibrium ones. The integration of Eq. (8) with the boundary conditions $(f/f_0)|_{p=0} = 1$ and $(f/f_0)|_{p=\pm 2\pi\hbar n} = 0$ yields

$$s_{\pm} = \left(\int_0^{\pm 2\pi\hbar n} \frac{dp}{B_{\pm} f_{0,\pm}}\right)^{-1}. \tag{9}$$

Here we imply the case of small temperatures $T \ll \Delta E$ and small net velocities $v \ll c$. Then in the leading order the integral [Eq. (9)] is evaluated as

$$s_{\pm} = \pm\frac{B_{0,\pm}}{4\hbar n}\left(\frac{2\hbar nc}{\pi T}\right)^{1/2}\exp\left(-\frac{4\hbar nc}{3T} \mp \frac{\pi\hbar n v_{\pm}}{T}\right), \tag{10}$$

where $B_{0,\pm}$ is the "diffusion" coefficient $B_{\pm}$ at $u=0$. Using the exact form of f_0 one finds that $B_{\pm} = -A_{\pm}T/(u+v)$. The coefficient $A = dp/dt$ is just the viscous friction force acting on solitons. At small velocities this force is proportional to the velocity of the soliton motion relative the normal component $A = -\eta(u+v)$ that yields $B = \eta T$, where η is the friction coefficient (under assumption that η is the same for "+" and "–" solitons, the coefficients $A_{\pm} = A$ and $B_{\pm} = B$ are the same as well).

The soliton distribution functions are normalized by the condition $n_{sol} = (1/2\pi\hbar)\int f dp$, where n_{sol} is the soliton density. The quantities $s_{\pm}$ are the fluxes in the direction of larger momenta p (negative sign of s_- means that actual direction of the flux is the opposite one). The difference of their modules determines the frequency of PS's:

$$\nu = \frac{(|s_-| - s_+)l}{2\pi\hbar}. \tag{11}$$

Using Eqs. (10) and (11) one finds the parameter α and obtains the following expression for the decay time:

$$\tau = \tau_0 \frac{1}{n_v l^2}\left(\frac{T}{2\pi\hbar nc}\right)^{1/2}\exp\left(\frac{4\hbar nc}{3T}\right), \tag{12}$$

where $\tau_0 = m/\eta$. According to Eq. (12) the crossover temperature is $T_0 = 4\hbar nc/3$. For weakly nonideal Bose gas $T_0/T_d \sim \sqrt{\gamma n/T_d} \ll 1$ and $T_0/T_c \sim l/\xi \gg 1$ so $T_c \ll T_0 \ll T_d$.

To evaluate the parameter τ_0 we consider the friction connected with the interaction of the solitons with phonons. The dark soliton is the exact solution of the 1D GP equation, and it does not interact with phonons in 1D. But the dislocation core is not a strict 1D system. It is a quasi-1D system with a small but finite cross section. Such a system is described by an effective 1D GP equation with an additional higher order in Ψ interaction term:[27] $i\hbar\partial\Psi/\partial t = -(\hbar^2/2m)\partial^2\Psi/\partial x^2 + \gamma|\Psi|^2\Psi - \gamma_1|\Psi|^4\Psi$. Due to such a term the reflection coefficient $R \sim (\gamma_1 n/\gamma)^2$ for the phonons that scatter on soliton is nonzero.[27]

The parameter $\gamma_1 n$ is evaluated as $\gamma_1 n \sim \gamma(r_{\perp}/\xi)^2$, where $r_{\perp}$ is the radius the supefluid channel. As was shown in Ref. 27, at small u (and $v=0$) the time derivative of the soliton momentum is given by the expression $\dot{p} = -\eta u = -C(mRT/\hbar)u$, where the numerical factor $C \sim 1$. It yields $\tau_0 \sim (\hbar/T)(\xi/r_{\perp})^4$. The scale of τ_0 is determined by the quantity $\hbar/T$ that is of order of 10^{-10} c^{-1} for $T=0.1$ K.

It is necessary to note one important point. One could think that since the GP equation we use is invariant with respect to translations it is impossible to describe the relaxation of superflow in the GP approach. But our approach is basically the same as commonly used for the computation of forces that act on a vortex in a 3D superfluid caused by its interaction with phonons and rotons (see, for instance, Ref. 28). In the latter case the GP approach is applied for the computation of amplitudes of scattering of phonons on vortices and for finding the momentum flux over the cylindrical surface around the vortex line. In such a way one can obtain the rate of transfer of the momentum from a vortex to phonons, i.e., to the normal component, which is assumed to be in equilibrium with the environment (walls, substrate, etc.)

The specific of the 1D system is that for the GP equation with only cubic interaction term the phonons do not interact with solitons, and there is no transfer of the momentum from solitons to the normal component. The fifth-order term switches on that interaction and the momentum transfers from solitons to 1D phonons of the network. 1D phonons

interact with bulk phonons, and due to such an interaction the normal component in the network remains in equilibrium with the crystal [since the relaxation time in the phonon subsystem is much smaller than τ given by Eq. (12)]. Eventually, the momentum obtained from solitons is transferred to the crystal. Here we do not describe explicitly the mechanism of such a transfer but just imply that the normal component is in equilibrium with the environment.

Estimating Eq. (12) we obtain that the decay time is of order of few seconds at $T \approx 0.04T_0$, and it is of order of an hour at $T \approx 0.03T_0$. We note that at very small temperatures quantum jumps between states with different vortex configurations may become important (for 1D rings the quantum jumps were studied in Refs. 29 and 30). We estimate the quantum correction to the τ is $\tau_0 \propto e^{n\xi}$, and this correction becomes important at $T \lesssim \gamma n = T_0/n\xi \ll T_0$. The quantum correction results only in a modification of the law of increase in the relaxation time under lowering of temperature, and its accounting should not change the main conclusion on the emergence of the quasisuperfluid state in the network well above the critical temperature.

In this Rapid Communication we have considered quasisuperfluidity in a 2D network. The situation in a 3D network should be qualitatively the same. In the latter case superflow may decay due to expansion or shrinking of vortex rings. The mechanism of expansion (shrinking) of vortex rings in a 3D network is basically the same as vortex motion in a 2D network: both of them are connected with the phase slips in segments of the network.

In conclusion, we note that the results obtained can be also applied to multiple connected Bose-Einstein condensates of rarefied alkali gases in optical lattices,[31] where the quasisuperfluid state can be observed directly.

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